# Markov Chains

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Supplement to Modeling

# Discrete Markov Chains

- Given is a discrete state space for a system which is observed over time
- We denote the states with $i$, $j$,…and denote transition probabilities in the time $[0, t]$by

$$P(X_t = i \mid X_0 = j) = P_{ij}(t)$$

- The transition probabilities define a matrix (Transition matrix)
- We are interested in the behavior of the system after a long time

# Discrete Markov Chains

- The chain is called a homogeneous chain if the transition probabilities are the same for all times. Transition Matrix:

$$P(X_t = i \mid X_{t-1} = j) = P_{ij} \text{ und } P_{ij}(t) = \left(P^n\right)_{ij}$$

- For an initial distribution of the states

$$\pi(.) = \left\{\pi(1), \pi(2), \ldots\right\}$$

we obtain the probabilities of the states at time $t$ by $\pi \cdot P_{ij}(t)$

# Discrete Markov Chains

- The recurrence time for a state $i$ is defined by

$$\tau_{ii} = \min\{t > 0 : X_t = i \mid X_0 = i\}$$

- The Markov chain is irreducible, if for all $i$, $j$ exists a $t$, such that $P_{ij}(t) > 0$
- An irreducible Markov chain is recurrent, if

$$P(\tau_{ii} < \infty) = 1$$

or equivalent $\sum_t P_{ij}(t) = \infty$

# Discrete Markov Chains

- An irreducible Markov chain is positive recurrent if $E(\tau_{ii}) < \infty$ for all $i$, and zero-recurrent otherwise
- An irreducible Markov chain is positive recurrent if there exists a stationary distribution for the states such that for all $i, j$ the following equation holds:

$$\sum_i \pi(i) P_{ij}(t) = \pi(j)$$

# Discrete Markov Chains

- An irreducible Markov chain is called aperiodic, if for all states $i$ the 1 is the largest common divisor of the sets

$$\{t > 0 : t = 1,2,\ldots, P_{ii}(t) > 0\}$$

- For a positive recurrent and aperiodic Markov chain exist always a unique stationary distribution defined by the equation

$$\sum_i \pi(i) P_{ij}(t) = \pi(j)$$

# Discrete Markov Chains

- A positive recurrent and aperiodic Markov chain is called ergodic Markov chain
- The following propositions hold:

$$P_{ij}(t) \to \pi(j) \quad \text{for all } \ i, j$$

$$\text{If} \quad E(|f(X)|) < \infty \Rightarrow P(\bar{f}_N \to E(f(X))) = 1$$

$$\bar{f}_N = \frac{1}{N} \sum_{t=1}^{N} f(X_t), \quad X_t \text{ generated by the chain}$$

# Discrete Markov Chains

- A positive recurrent Markov chain wit stationary distribution is called reversible if

$$\pi(i)P_{ij} = \pi(j)P_{ji} \quad \text{for all } i, j$$

(Detailed balance equation)