Partielle Ableitungen - Teil 1

6) Gebe die partiellen Ableitungen für die folgenden Funktionen an

- $f(x,y) = \sin(x)e^{xy}$

$$\frac{\partial f(x,y)}{\partial x} = \cos(x)e^{xy} + y\sin(x)e^{xy}$$

$$\frac{\partial f(x,y)}{\partial y} = \sin(x)\,x\,e^{xy}$$

- $f(x,y) = \frac{x^2y + x}{y}$

$$\frac{\partial f(x,y)}{\partial x} = \frac{(2xy+1)y - (x^2y+x)0}{y^2} = \frac{2xy+1}{y^2}y = \frac{2xy+1}{y}$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{x^2 \cdot y - (x^2y+x)1}{y^2} = \frac{-x}{y^2}$$

- $f(x,y) = \sin(\cos(x)) + 5y^2$

$$\frac{\partial f(x,y)}{\partial x} = \cos(\cos(x)) \cdot (-\sin(x))$$

$$\frac{\partial f(x,y)}{\partial y} = 10y$$

- $f(x,y,z) = e^{xyz^2}$

$$\frac{\partial f(x,y,z)}{\partial x} = e^{xyz^2} \cdot yz^2$$

$$\frac{\partial f(x,y,z)}{\partial y} = e^{xyz^2} \cdot xz^2$$

$$\frac{\partial f(x,y,z)}{\partial z} = e^{xyz^2} \cdot xyz \cdot 2$$

- $f(x,y) = \sin(xy)$

$$\frac{\partial f(x,y)}{\partial x} = \cos(xy) \cdot y$$

$$\frac{\partial f(x,y)}{\partial y} = \cos(xy) \cdot x$$

7) Gebe die Definitionsbereiche folgender Funktionen an

- $f(x,y) = \frac{y}{\sin(x)}$

  $D_f = \{(x,y) \in \mathbb{R}^2 \mid \sin(x) \neq 0\} = \{(x,y) \mid x \neq n\pi \;\; \forall n \in \mathbb{Z}\}$

- $f(x,y) = \frac{3}{x+y}$

  $D_f = \{(x,y) \in \mathbb{R}^2 \mid x+y \neq 0\} = \{(x,y) \mid x \neq -y\}$

- $f(x,y) = xy$

  $D_f = \{(x,y) \in \mathbb{R}^2\}$

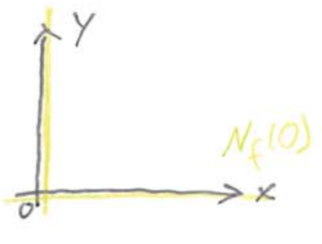
- $f(x) = \log(x)$

  $D_f = \{x \geq 0\}$

8) Gebe die Niveaumengen $N_f(0)$ an und zeichne sie

- $f(x,y) = x^2 y^3$

  $N_f(0) = \{(x,y) \mid f(x,y) = 0\} = \{(x,y) \mid x^2 y^3 = 0\} = \{(x,y) \mid x = 0 \vee y = 0\}$



9) Gebe die Gradienten folgender Funktionen an

- $f(x,y) = \sin(x) e^{xy}$

  $\operatorname{grad} f = \begin{pmatrix} \cos(x) e^{xy} + y \sin(x) e^{xy} \\ \sin(x) x e^{xy} \end{pmatrix}$

- $f(x,y) = \frac{x^2 y + x}{y}$

  $\operatorname{grad} f = \begin{pmatrix} \frac{2xy+1}{y} \\ \frac{-x}{y^2} \end{pmatrix}$

- $f(x,y) = 5xy + x^2 + 3y$

$\frac{\partial f}{\partial x} = 5y + 2x \, , \quad \frac{\partial f}{\partial y} = 5x + 3$

$\operatorname{grad} f = \begin{pmatrix} 5y + 2x \\ 5x + 3 \end{pmatrix}$

- $f(x,y) = e^{3x} + 2xy$

$\frac{\partial f}{\partial x} = 3e^{3x} + 2 \quad \frac{\partial f}{\partial y} = 2x$

$\operatorname{grad} f = \begin{pmatrix} 3e^{3x} + 2y \\ 2x \end{pmatrix}$

10) Gebe die Hessematrix der folgenden Funktionen an

- $f(x,y) = \frac{x^2y + x}{y}$

$\frac{\partial^2 f}{\partial x \partial x} = \frac{2y}{y} = 2 \, , \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{2xy - 2xy - 1}{y^2} = \frac{-1}{y^2} \, , \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{-1}{y^2} \, , \quad \frac{\partial^2 f}{\partial y \partial y} = \frac{2x}{y^3}$

$Hf = \begin{pmatrix} \frac{\partial^2 f}{\partial x \partial x} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y \partial y} \end{pmatrix} = \begin{pmatrix} 2 & \frac{-1}{y^2} \\ \frac{-1}{y^2} & \frac{2x}{y^3} \end{pmatrix}$

- $f(x,y) = 5xy + x^2 + 3y$

$\frac{\partial^2 f}{\partial x \partial x} = 2 \, , \quad \frac{\partial^2 f}{\partial x \partial y} = 5 \, , \quad \frac{\partial^2 f}{\partial y \partial x} = 5 \, , \quad \frac{\partial^2 f}{\partial y \partial y} = 0$

$H_f = \begin{pmatrix} 2 & 5 \\ 5 & 0 \end{pmatrix}$

- $f(x,y) = e^{3x} + 2xy$

$\frac{\partial^2 f}{\partial x \partial x} = 9e^{3x} \, , \quad \frac{\partial^2 f}{\partial x \partial y} = 2 \, , \quad \frac{\partial^2 f}{\partial y \partial x} = 2 \, , \quad \frac{\partial^2 f}{\partial y \partial y} = 0$

$H_f = \begin{pmatrix} 9e^{3x} & 2 \\ 2 & 0 \end{pmatrix}$

11) Gebe die Jacobimatrizen der folgenden Funktionen an

- $f(x,y) = \begin{pmatrix} 5x^2 + \sin(xy) \\ 10x + 2y \end{pmatrix}$

$$J_f = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 10x + y\cos(xy) & x\cos(xy) \\ 10 & 2 \end{pmatrix}$$

- $f(x,y,z) = \begin{pmatrix} x^2 + y \cdot z \cdot 3 + 5z^2x \\ 10zxy + 3xy \end{pmatrix}$

$$J_f = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_1}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x + 5z^2 & 3z & 3y + 10xz \\ 10yz + 3y & 10xz + 3x & 10xy \end{pmatrix}$$

- $f(x,y) = \begin{pmatrix} \sin(xy) \\ \cos(xy) \\ xy \end{pmatrix}$

$$J_f = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{pmatrix} = \begin{pmatrix} y\cos(xy) & x\cos(xy) \\ -y\sin(xy) & -x\sin(xy) \\ y & x \end{pmatrix}$$

12) Gebe die Richtungsableitungen für f im Punkt P in Richtung v an

- $f(x,y,z) = x^2y - 4xyz \qquad P = (1,-1,2) \quad v = (1,0,1)$

$$\operatorname{grad} f = \begin{pmatrix} 2xy - 4yz \\ x^2 - 4xz \\ -4xy \end{pmatrix}$$

$$D_v f = \operatorname{grad} f \cdot \frac{v}{\|v\|} = \begin{pmatrix} 2xy - 4yz \\ x^2 - 4xz \\ -4xy \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{1^2+0^2+1^2}}$$

$$= \frac{1}{\sqrt{2}}(2xy - 4yz - 4xy)$$

$$D_v f(1,-1,2) = \frac{1}{\sqrt{2}} \left(2\cdot 1\cdot(-1) - 4\cdot(-1)\cdot 2 - 4\cdot 1\cdot(-1)\right)$$

$$= \frac{1}{\sqrt{2}} \cdot 10 = \sqrt{2}\cdot 5$$

- $f(x,y,z) = x^3y^2 - 5x^2 + 10z \qquad P=(2,3,1) \qquad v = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$

$$\text{grad } f = \begin{pmatrix} 3x^2y^2 - 10x \\ 2yx^3 \\ 10 \end{pmatrix}$$

$$D_v f = \text{grad} f \bullet \frac{v}{\|v\|} = \begin{pmatrix} 3x^2y^2 - 10x \\ 2yx^3 \\ 10 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{3^2+4^2+0^2}}$$

$$= \frac{1}{\sqrt{25}} \left(3\cdot(3x^2y^2 - 10x) + 4\cdot 2yx^3\right)$$

$$D_v f(2,3,1) = \frac{1}{\sqrt{25}} \left(3\cdot(3\cdot 2^2\cdot 3^2 - 10\cdot 2) + 8\cdot 3\cdot 2^3\right)$$

$$= \frac{1}{5}(324 - 60 + 192) = \frac{456}{5} = 91{,}2$$