Mehrfachintegrale

9) Berechne die folgenden Integrale

- $\int_M e^{-x} \cdot y \, d(x,y) \qquad M = \{(x,y) \mid 0 \le x \le \infty, 2 \le y \le 4\}$

$$= \int_2^4 \int_0^\infty e^{-x} \cdot y \, dx dy = \int_2^4 [-ye^{-x}]_{x=0}^\infty \, dy = \int_2^4 0 - (-ye^{-0}) \, dy = \int_2^4 y \, dy$$

$$= [\tfrac{1}{2} y^2]_{y=2}^4 = \tfrac{1}{2} 4^2 - \tfrac{1}{2} 2^2 = \tfrac{16}{2} - \tfrac{4}{2} = 6$$

- $\int_M x \cdot y + x^2 + z \, d(x,y,z) \qquad M = \{(x,y,z) \mid y \le x \le z, 0 \le y \le 1, 0 \le z \le 2y\}$

$$= \int_0^1 \int_0^{2y} \int_y^z xy + x^2 + z \, dx dz dy = \int_0^1 \int_0^{2y} [\tfrac{1}{2} x^2 y + \tfrac{1}{3} x^3 + zx]_{x=y}^z \, dz dy$$

$$= \int_0^1 \int_0^{2y} \tfrac{1}{2} z^2 y + \tfrac{1}{3} z^3 + z^2 - \tfrac{1}{2} y^3 - \tfrac{1}{3} y^3 - zy \, dz dy = \int_0^1 [\tfrac{1}{12} z^4 + \tfrac{1}{6} z^3 y + \tfrac{1}{3} z^3 - \tfrac{5}{6} z y^3 - \tfrac{z^2}{2} y]_{z=0}^{2y} dy$$

$$= \int_0^1 (\tfrac{1}{12}(2y)^4 + \tfrac{1}{6}(2y)^3 y + \tfrac{1}{3}(2y)^3 - \tfrac{5}{6}(2y) y^3 - \tfrac{1}{2}(2y)^2 y \, dy = \int_0^1 \tfrac{16}{12} y^4 + \tfrac{8}{6} y^4 + \tfrac{8}{3} y^3 - \tfrac{10}{6} y^4 - \tfrac{4}{2} y^3 dy$$

$$= \int_0^1 y^4 + \tfrac{2}{3} y^3 dy = [\tfrac{1}{5} y^5 + \tfrac{2}{3} \tfrac{1}{4} y^4]_0^1 = \tfrac{1}{5} + \tfrac{2}{12} = \tfrac{11}{30}$$

- $\int_M \sin(xy) \, d(x,y) \qquad M = \{(x,y) \mid 0 \le x^2 \le y^2, 0 \le y \le 4\}$

$M = \{(x,y) \mid -y \le x \le y, 0 \le y \le 4\}$

$$\int_M \sin(\tfrac{x}{y}) \, d(x,y) = \int_0^4 \int_{-y}^y \sin(\tfrac{x}{y}) \, dx dy = \int_0^4 [y \cos(\tfrac{x}{y})]_{-y}^y \, dy$$

$$= \int_0^4 -y \cos(\tfrac{y}{y}) - y(-\cos(-\tfrac{y}{y})) \, dy \overset{\cos(-x) = \cos(x)}{=} \int_0^4 -y \cos(1) + y \cos(1) \, dy$$

$$= \int_0^4 0 \, dy = 0$$

10) Berechne die folgenden Integrale mittels Transformationen

- Berechne die Fläche von $M=\{(x,y) \mid x^2+y^2 \leq a\}$ mittels Kreiskoordinaten

Kreiskoordinaten: $x=r\cdot\cos(\varphi)$, $y=r\cdot\sin(\varphi)$

$\Rightarrow r=\sqrt{x^2+y^2}$

$$D=\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \cos(\varphi) & -r\sin(\varphi) \\ \sin(\varphi) & r\cdot\cos(\varphi) \end{pmatrix}$$

$$\Rightarrow |\det D| = |(\cos(\varphi)\cdot r\cdot\cos(\varphi)) - (-r\cdot\sin(\varphi)\cdot\sin(\varphi))|$$

$$= |r\cos^2(\varphi)+r\sin^2(\varphi)| = |r\cdot(\cos^2(\varphi)+\sin^2(\varphi))|$$

$$= |r| = r \quad \text{da } = \sqrt{x^2+y^2} \geq 0$$

$$\tilde{M} = \{(r,\varphi) \mid r^2 \leq a\} \cap \underbrace{\{(r,\varphi) \mid r\geq 0,\ 0\leq\varphi\leq 2\pi\}}_{\text{da } r \text{ Radius, } \varphi \text{ Winkel}} = \{(r,\varphi) \mid 0\leq r\leq \sqrt{a},\ 0\leq\varphi\leq 2\pi\}$$

$$\iint_M 1\,dx\,dy = \iint_{\tilde{M}} 1\cdot|\det D|\,dr\,d\varphi = \int_0^{2\pi}\int_0^{\sqrt{a}} 1\cdot r\,dr\,d\varphi$$

$$= \int_0^{2\pi} \left[\tfrac{1}{2}r^2\right]_0^{\sqrt{a}} d\varphi = \int_0^{2\pi} \tfrac{1}{2}a - 0\,d\varphi = \left[\tfrac{a}{2}\varphi\right]_0^{2\pi}$$

$$= \tfrac{a}{2}2\pi - 0 = a\pi$$

- Berechne $\iint_M x-3y\,dx\,dy$ mit $M=\{(x,y) \mid y\geq \frac{1}{2}x,\ y\leq 2x,\ y\leq 3-x\}$ mithilfe der Transformationen $x=2u+v$, $y=u+2v$

$$\tilde{M} = \{(u,v) \mid u+2v \geq \tfrac{2u+v}{2},\ u+2v\leq 2(2u+v),\ u+2v\leq 3-(2u+v)\}$$

$$= \{(u,v) \mid u+2v\geq u+\tfrac{v}{2},\ u+2v\leq 4u+2v,\ u+2v\leq 3-2u-v\}$$

$$= \{(u,v) \mid \tfrac{3}{2}v\geq 0,\ 0\leq 3u,\ 3u+3v\leq 3\} = \{(u,v) \mid v\geq 0,\ u\geq 0,\ u+v\leq 1\}$$

$$= \{(u,v) \mid 0\leq v\leq 1,\ 0\leq u\leq 1-v\}$$

$f(x,y) = x - 3y$

$\tilde{f}(u,v) = 2u + v - 3(u+2v) = 2u + v - 3u - 6v = -u - 5v$

$$D = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial(2u+v)}{\partial u} & \frac{\partial(2u+v)}{\partial v} \\ \frac{\partial(u+2v)}{\partial u} & \frac{\partial(u+2v)}{\partial v} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$|\det D| = |2\cdot 2 - 1\cdot 1| = |3| = 3$

$$\iint_M x - 3y \, dx\,dy = \iint_{\hat{M}} \tilde{f}(u,v) \cdot |\det D| \, du\,dv$$

$$= \int_0^1 \int_0^{1-v} (-u - 5v) \cdot 3 \, du\,dv$$

$$= -3 \int_0^1 \int_0^{1-v} u + 5v \, du\,dv$$

$$= -3 \int_0^1 \left[\tfrac{1}{2}u^2 + 5vu\right]_0^{1-v} dv$$

$$= -3 \int_0^1 \tfrac{1}{2}(1-v)^2 + 5v(1-v) - 0 \, dv$$

$$= -3 \int_0^1 \tfrac{1}{2} - v + \tfrac{1}{2}v^2 + 5v - 5v^2 \, dv$$

$$= -3 \int_0^1 \tfrac{1}{2} + 4v - \tfrac{9}{2}v^2 \, dv$$

$$= -3 \left[\tfrac{1}{2}v + 2v^2 - \tfrac{9}{6}v^3\right]_0^1$$

$$= -3\left(\left(\tfrac{1}{2} + 2 - \tfrac{3}{2}\right) - 0\right) = -3$$