Aufgaben zum Tutorium 1 – MG2

Differentialrechnung

1) Leite die folgenden Funktionen ab:

- $f(x) = \frac{1}{x^4}$

  $f'(x) = (x^{-4})' \overset{\text{Potenzregel}}{=} -4x^{-5} = \frac{-4}{x^5}$

- $f(x) = (x^2+1)(x^3+1)$

  Variante 1: $f'(x) = (x^5+x^3+x^2+1)' \overset{\text{Potenz-, Additionsreg.}}{=} 5x^4 + 3x^2 + 2x$

  Variante 2: $f'(x) \overset{\text{Produktregel}}{=} (x^2+1)'(x^3+1) + (x^2+1)(x^3+1)' = 2x(x^3+1) + (x^2+1)3x^2$

  $= 2x^4 + 2x + 3x^4 + 3x^2 = 5x^4 + 3x^2 + 2x$

- $f(x) = \frac{x^2+3}{x}$

  $f'(x) \overset{\text{Quotientenregel}}{=} \frac{(x^2+3)' \cdot x - (x^2+3)(x)'}{x^2} = \frac{2x \cdot x - (x^2+3)}{x^2} = \frac{2x^2 - x^2 - 3}{x^2} = \frac{x^2-3}{x^2}$

- $f(x) = \sqrt[3]{x+5x^2}$ $\quad g(y) = \sqrt[3]{y}$ $\quad g'(y) = \frac{1}{3}(y)^{-2/3}$, $h(x) = x + 5x^2$

  $f'(x) = ((x+5x^2)^{\frac{1}{3}})' \overset{\text{Kettenregel}}{=} \frac{1}{3}(x+5x^2)^{-\frac{2}{3}}(x+5x^2)' = \frac{1}{3}(x+5x^2)^{-2/3} \cdot (10x+1)$

  $= \frac{10x+1}{3\sqrt[3]{x+5x^2}^{\,2}}$

- $f(x) = \sin(\cos(x) + x^4)$ $\quad g(y) = \sin(y)$, $g'(y) = \cos(y)$, $h(x) = \cos(x) + x^4$, $h'(x) = -\sin(x) + 4x^3$

  $f'(x) \overset{\text{Kettenregel}}{=} \cos(\cos(x) + x^4)(-\sin(x) + 4x^3)$

- $f(x) = e^{5x^4 + 2\cos(x) + \sin(x)}$ $\quad g(y) = e^y$, $g'(y) = e^y$, $h(x) = 5x^4 + 2\cos(x) + \sin(x)$

  $f'(x) \overset{\text{Kettenregel}}{=} e^{5x^4 + 2\cos(x) + \sin(x)} \cdot (5x^4 + 2\cos(x) + \sin(x))'$

  $= e^{5x^4 + 2\cos(x) + \sin(x)} \cdot (20x^3 - 2\sin(x) + \cos(x))$

2) Finde y' durch implizite Differentiation:

- $x^2 + yx + 5y = 0 \quad \Rightarrow \quad F(x, y(x)) = x^2 + y(x)x + 5y$

$$\frac{\partial F}{\partial x} = 2x + y, \quad \frac{\partial F}{\partial y} = x+5$$

$$\Rightarrow y' = \frac{-\partial F/\partial x}{\partial F/\partial y} = \frac{-2x-y}{x+5}$$

- $x^2 + 5y + \sin(x) = 0 \quad \Rightarrow \quad F(x, y(x)) = x^2 + 5y(x) + \sin(x)$

$$\frac{\partial F}{\partial x} = 2x + \cos(x), \quad \frac{\partial F}{\partial y} = 5$$

$$\Rightarrow y' = \frac{-\partial F/\partial x}{\partial F/\partial y} = \frac{-2x-\cos(x)}{5}$$

- $e^x + 3y = 2$

$\Leftrightarrow e^x + 3y - 2 = 0 \quad \Rightarrow F(x, y(x)) = e^x + 3y - 2$

$$\frac{\partial F}{\partial x} = e^x, \quad \frac{\partial F}{\partial y} = 3$$

$$\Rightarrow y' = \frac{-\partial F/\partial x}{\partial F/\partial y} = -\frac{1}{3}e^x$$

3) Ermittel y' durch Logarithmieren und Ableiten:

- $y = x^{x^2}$

$$y = x^{x^2} = e^{x^2 \cdot \log(x)}$$

$$\Leftrightarrow \log(y) = \log(e^{x^2 \cdot \log(x)}) = x^2 \cdot \log(x)$$

$$\Rightarrow \underbrace{\frac{d(\log(y))}{dx}}_{\frac{1}{y} \cdot y'} = (x^2 \cdot \log(x))' = 2x\log(x) + x^2 \cdot \frac{1}{x} = 2x \cdot \log(x) + x$$

$$\frac{1}{y} y' = 2x \log(x) + x$$

$$\Leftrightarrow y' = (2x \log(x) + x) \cdot y$$

$$= (2x \log(x) + x) \cdot x^{x^2}$$

$$x^a \cdot x = x^{a+1}$$

$$= (2 \log(x) + 1) \cdot x^{x^2+1}$$

- $y = \sin(x)^{\cos(x)}$

$$y = \sin(x)^{\cos(x)} = e^{\cos(x) \cdot \log(\sin(x))}$$

$$\Leftrightarrow \log(y) = \log(e^{\cos(x) \cdot \log(\sin(x))}) = \cos(x) \cdot \log(\sin(x))$$

$$\Rightarrow \underbrace{\frac{d(\log(y))}{dx}}_{= \frac{1}{y} \cdot y'} = (\cos(x) \cdot \log(\sin(x)))' = -\sin(x) \cdot \log(\sin(x)) + \frac{\cos^2(x)}{\sin(x)}$$

$$\Leftrightarrow y' = \left(-\sin(x) \cdot \log(\sin(x)) + \frac{\cos^2(x)}{\sin(x)}\right) \cdot y$$

$$= \left(-\sin(x) \cdot \log(\sin(x)) + \frac{\cos^2(x)}{\sin(x)}\right) \cdot \sin(x)^{\cos(x)}$$

- $y = \cos(x)^{x^2}$

$$y = \cos(x)^{x^2} = e^{x^2 \cdot (\log(\cos(x))}$$

$$\Leftrightarrow \log(y) = x^2 \cdot \log(\cos(x))$$

$$\Rightarrow \underbrace{\frac{d((\log(y))}{dx}}_{= \frac{1}{y} \cdot y'} = (x^2 \cdot \log(\cos(x)))' = 2x \cdot \log(\cos(x)) - x^2 \frac{\sin(x)}{\cos(x)}$$

$$\Leftrightarrow y' = \left(2x \log(\cos(x)) - x^2 \frac{\sin(x)}{\cos(x)}\right) \cdot y$$

$$= \left(2x \log(\cos(x)) - x^2 \frac{\sin(x)}{\cos(x)}\right) \cdot \cos(x)^{x^2}$$

4) Bestimme für die nachfolgenden Funktionen $f'(x)$ und $\|f(x)\|$:

- $f(x) = \begin{pmatrix} x^5 \\ 2x+3 \\ \sin(x) \end{pmatrix}$

$$f'(x) = \begin{pmatrix} (x^5)' \\ (2x+3)' \\ (\sin(x))' \end{pmatrix} = \begin{pmatrix} 5x^4 \\ 2 \\ \cos(x) \end{pmatrix}$$

$$\|f(x)\| = \sqrt{(x^5)^2 + (2x+3)^2 + (\sin(x))^2} = \sqrt{x^{10} + 2x^2 + 6x + 9 + \sin^2(x)}$$

- $f(x) = \begin{pmatrix} \cos(x) \\ \sin(x) \\ 0 \end{pmatrix}$

$$f'(x) = \begin{pmatrix} (\cos(x))' \\ (\sin(x))' \\ (0)' \end{pmatrix} = \begin{pmatrix} -\sin(x) \\ \cos(x) \\ 0 \end{pmatrix}$$

$\cos(x)^2 + \sin(x)^2 = 1$

$$\|f(x)\| = \sqrt{(\cos(x))^2 + (\sin(x))^2 + 0^2} = \sqrt{\cos^2(x) + \sin^2(x)} = \sqrt{1} = 1$$

- $f(x) = \begin{pmatrix} e^{2x} \\ x^2 \\ -x^2 \end{pmatrix}$

$$f'(x) = \begin{pmatrix} (e^{2x})' \\ (x^2)' \\ (-x^2)' \end{pmatrix} = \begin{pmatrix} 2e^{2x} \\ 2x \\ -2x \end{pmatrix}$$

$$\|f(x)\| = \sqrt{(e^{2x})^2 + (x^2)^2 + (-x^2)^2} = \sqrt{e^{4x} + x^4 + x^4} = \sqrt{e^{4x} + 2x^4}$$

5) Berechne die folgenden Kreuzprodukte:

- $\begin{pmatrix} \sin(x) \\ \cos(x) \\ 2x \end{pmatrix} \times \begin{pmatrix} -\cos(x) \\ \sin(x) \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} \cos(x) \cdot 0 - 2x \cdot \sin(x) \\ 2x(-\cos(x)) - \sin(x) \cdot 0 \\ \sin(x) \cdot \sin(x) - \cos(x)(-\cos(x)) \end{pmatrix} = \begin{pmatrix} -2x\sin(x) \\ -2x\cos(x) \\ \sin^2(x) + \cos^2(x) \end{pmatrix} = \begin{pmatrix} -2x\sin(x) \\ -2x\cos(x) \\ 1 \end{pmatrix}$$

- $\begin{pmatrix} x^2 \\ 3x \\ 1 \end{pmatrix} \times \begin{pmatrix} 5x+1 \\ 2x \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} 3x \cdot 0 - 1 \cdot 2x \\ 1 \cdot (5x+1) - x^2 \cdot 0 \\ x^2 \cdot 2x - 3x(5x+1) \end{pmatrix} = \begin{pmatrix} -2x \\ 5x+1 \\ 2x^3 - 15x^2 - 3x \end{pmatrix}$$