Bestimmte Integrale

4) Berechne die Fläche zwischen den folgenden Funktionen im Intervall zwischen den reellwertigen Schnittpkten der Funktionen

- $f(x) = x^2,\ g(x) = x$

$f(x) = g(x) \Leftrightarrow x^2 = x \Leftrightarrow x = 1 \vee x = 0$

$$\left|\int_0^1 f(x) - g(x)\,dx\right| = \left|\int_0^1 x^2 - x\,dx\right| = \left|\int_0^1 x^2 dx - \int_0^1 x\,dx\right|$$

$$= \left|\left[\tfrac{1}{3}x^3\right]_0^1 - \left[\tfrac{1}{2}x^2\right]_0^1\right| = \left|\left(\tfrac{1}{3} - 0\right) - \left(\tfrac{1}{2} - 0\right)\right|$$

$$= \left|-\tfrac{1}{6}\right| = \tfrac{1}{6}$$

- $f(x) = x^2,\ g(x) = 2x^2 - 4$

$f(x) = g(x) \Leftrightarrow x^2 = 2x^2 - 4 \Leftrightarrow 0 = x^2 - 4 \Leftrightarrow x = -2 \vee x = 2$

$$\left|\int_{-2}^2 f(x) - g(x)\,dx\right| = \left|\int_{-2}^2 -x^2 + 4\,dx\right| = \left|\int_{-2}^2 -x^2 dx + \int_{-2}^2 4\,dx\right|$$

$$= \left|\left[\tfrac{-1}{3}x^3\right]_{-2}^2 + \left[4x\right]_{-2}^2\right| = \left|\left(\tfrac{-1}{3}\cdot 8 + \tfrac{-1}{3}8\right) + (4\cdot 2 + 4\cdot 2)\right|$$

$$= \left|\tfrac{-16}{3} + 16\right| = \left|\tfrac{32}{3}\right| = \tfrac{32}{3}$$

- $f(x) = 5,\ g(x) = x^2 - 4$

$f(x) = g(x) \Leftrightarrow 5 = x^2 - 4 \Leftrightarrow 0 = x^2 - 9 \Leftrightarrow x = -3 \vee x = 3$

$$\left|\int_{-3}^3 f(x) - g(x)\,dx\right| = \left|\int_{-3}^3 -x^2 + 9\,dx\right| = \left|\left[\tfrac{-1}{3}x^3\right]_{-3}^3 + \left[9x\right]_{-3}^3\right|$$

$$= \left|(-9 - 9) + (27 + 27)\right| = \left|-18 + 54\right| = |36| = 36$$

5) Berechne die folgenden Integrale

- $\int_0^5 y^2 dx$ für $y^2 = x^2 - 4$

$= \int_0^5 x^2 - 4\, dx = \left[\frac{1}{3}x^3 - 4x\right]_0^5 = \frac{1}{3}5^3 - 4 \cdot 5 - 0 = \frac{125}{3} - 20 = \frac{65}{3}$

- $\int_{-2}^2 y\, dx$ für $\sqrt{y} = e^x$

$\sqrt{y} = e^x \Leftrightarrow y = (e^x)^2 = e^{2x}$

$\int_{-2}^2 y\, dx = \int_{-2}^2 e^{2x} dx = \left[\frac{1}{2}e^{2x}\right]_{-2}^2 = \frac{1}{2}e^4 - \frac{1}{2}e^{-4} = \frac{1}{2}(e^4 - e^{-4})$

- $\int_0^r (y')^2 dx$ für $y = r \cdot x^2 + 10$

$y' = 2rx$

$\int_0^r (y')^2 dx = \int_0^r (2rx)^2 dx = \int_0^r 4r^2x^2 dx = \left[\frac{4}{3}r^2x^3\right]_0^r$

$= \frac{4}{3}r^2r^3 - 0 = \frac{4}{3}r^5$

- $\int_{\sqrt{6}}^3 x\sqrt{x^2 - 5}\, dx$ $\qquad z = x^2 - 5 \qquad dx = \frac{1}{2x} dz$

$\int x\sqrt{x^2-5}\, dx = \int x\sqrt{z}\, dx = \int \sqrt{z} \frac{x}{2x} dz = \frac{1}{2}\int \sqrt{z}\, dz$

$= \frac{1}{2}\frac{2}{3} z^{3/2} = \frac{1}{3} z^{3/2} = \frac{1}{3}(x^2 - 5)^{3/2}$

$\int_6^9 x\sqrt{x^2-5}\, dx = \left[\frac{1}{3}(x^2-5)^{3/2}\right]_{\sqrt{6}}^3 = \left(\frac{1}{3}(3^2-5)^{3/2}\right) - \left(\frac{1}{3}(\sqrt{6}^2-5)^{3/2}\right)$

$= \frac{1}{3}4^{3/2} - \frac{1}{3}1^{3/2} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$